Some Proofs About Fibonacci Numbers

Nitin Verma
mathsanew.com

May 27, 2020

The *Fibonacci Sequence* is defined as:

$$
\begin{aligned}
&F_1 = 1 \\
&F_2 = 1 \\
&F_n = F_{n-2} + F_{n-1} \quad \forall n > 2
\end{aligned} \tag{1}
$$

So its first few terms are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377

The very simple recurrence definition of the Fibonacci Sequence gives rise to many simple-structured relations among the Fibonacci Numbers. In this article we explore some of them.

All the indices and numbers referred in this article are non-negative integers unless otherwise stated.

Relation 1

Since $F_{n+1} = F_{n-1} + F_n$, so if we take product $F_n \cdot F_{n+1}$, the outcome will again contain a similar and smaller product term: $F_{n-1} \cdot F_n$. We can attempt to repeat this process for subsequent smaller product terms. Assume n to be large enough to allow some iterations and observation of the repeating pattern.

$$\begin{aligned}
F_nF_{n+1} &= F_n(F_{n-1} + F_n) \\
&= F_{n-1}F_n + {F_n}^2 \\
&= F_{n-1}(F_{n-2} + F_{n-1}) + {F_n}^2 \\
&= F_{n-2}F_{n-1} + {F_{n-1}}^2 + {F_n}^2 \\
&= F_{n-2}(F_{n-3} + F_{n-2}) + {F_{n-1}}^2 + {F_n}^2 \\
&= F_{n-3}F_{n-2} + {F_{n-2}}^2 + {F_{n-1}}^2 + {F_n}^2 \\
&\quad \{\text{keep repeating for subsequent product terms}\} \\
&= \dots \\
&= F_1F_2 + {F_2}^2 + {F_3}^2 + \dots + {F_{n-1}}^2 + {F_n}^2
\end{aligned}$$

Since $F_1F_2 = 1$, we can write below relation for all $n \geq 2$:

$$F_nF_{n+1} = 1 + \sum_{k=2}^{n} {F_k}^2 \tag{2}$$

Example: $F_6F_7 = 8 \cdot 13 = 104 = 1 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2$

Relation 2

We can obtain below from the recurrence definition (1) of F_n. Again, assume n to be large enough to allow some iterations.

$$F_n = F_{n-2} + F_{n-1} = (F_{n-2} + F_{n-3}) + F_{n-2} = (F_{n-2} + F_{n-3} + F_{n-4}) + F_{n-3}$$

In above, we start seeing that the second smallest fibonacci term is occurring twice. We can pick one of the two occurrences of such term and expand it using its recurrence definition, and repeat as below:

$$\begin{aligned}
F_n &= (F_{n-2} + F_{n-3} + F_{n-4}) + F_{n-3} \\
&\quad \{\text{expanding } F_{n-3}\} \\
&= (F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5}) + F_{n-4} \\
&\quad \{\text{repeating as above}\} \\
&= (F_{n-2} + F_{n-3} + \dots + F_{n-k+1} + F_{n-k}) + F_{n-k+1} \\
&\quad 3 \leq k \leq n-1 \\
&\quad \{\text{with } k = n-1\} \\
&= (F_{n-2} + F_{n-3} + \dots + F_2 + F_1) + F_2
\end{aligned}$$

Since $F_2 = 1$, we can write below relation for all $n > 2$:

$$F_n = (\sum_{i=1}^{n-2} F_i) + 1 \tag{3}$$

Example: $F_9 = 34 = (1 + 1 + 2 + 3 + 5 + 8 + 13) + 1$

Relation 3

Below we will start with the recurrence definition (1) of F_n and repeatedly expand the higher index fibonacci number present. Assume n to be large enough to allow some iterations.

$$\begin{aligned}
F_n &= F_{n-2} + F_{n-1} \\
&\quad \{\text{expanding } F_{n-1}\} \\
&= F_{n-2} + (F_{n-3} + F_{n-2}) \\
&= F_{n-3} + (2)F_{n-2} \\
&\quad \{\text{expanding } F_{n-2}\} \\
&= F_{n-3} + (2)(F_{n-4} + F_{n-3}) \\
&= (2)F_{n-4} + (3)F_{n-3} \\
&\quad \{\text{repeatedly expanding as above}\} \\
&= (3)F_{n-5} + (5)F_{n-4} \\
&= (5)F_{n-6} + (8)F_{n-5}
\end{aligned}$$

At each step, from $aF_x + bF_{x+1}$, we get:

$$aF_x + b(F_{x-1} + F_x) = bF_{x-1} + (a + b)F_x$$

Interestingly, the multipliers of the fibonacci numbers are forming below sequence from one step to the next:

$$(1, 1), (1, 2), (2, 3), (3, 5), (5, 8), \ldots, (a, b), (b, a + b), (a + b, a + 2b)$$

where both terms in the pairs are forming a part of the fibonacci sequence. So, we can write:

$$
\begin{aligned}
F_n &= (1)F_{n-2} + (1)F_{n-1} \\
&= (F_1)F_{n-2} + (F_2)F_{n-1} \\
&= (F_2)F_{n-3} + (F_3)F_{n-2} \\
&= (F_3)F_{n-4} + (F_4)F_{n-3} \\
&= (F_4)F_{n-5} + (F_5)F_{n-4} \\
&= \ldots \\
&= (F_k)F_{n-k-1} + (F_{k+1})F_{n-k} \qquad 1 \leq k \leq (n-2)
\end{aligned}
$$

So, we can write below relation for all $n > 2$:

$$F_n = (F_k)F_{n-k-1} + (F_{k+1})F_{n-k} \qquad 1 \leq k \leq (n-2) \tag{4}$$

Note that for $k = n-2$, we are back to the original recurrence definition of F_n:

$$F_n = (F_{n-2})F_1 + (F_{n-1})F_2 = F_{n-2} + F_{n-1}$$

While the recurrence definition of F_n in (1) relates it only to the two preceding fibonacci numbers (F_{n-2}, F_{n-1}), the equation (4) relates it to other lower indices fibonacci numbers and will be very useful in finding other relations about fibonacci numbers.

Example: for $n = 10$ and $k = 6$, $F_{10} = 55 = F_6F_3 + F_7F_4 = 8 \cdot 2 + 13 \cdot 3$

Relation 4

We saw in equation (4) how a fibonacci number F_n could be expressed in terms of four other smaller fibonacci numbers. Notice that as the index k increases from 1 to $n-2$, the other index $n-k$ decreases from $n-1$ to 2. So we can try to choose some k for which some of these four indices $(k, k+1, n-k-1, n-k)$ turn out to be the same.

To make k and $n-k$ as equal, we need $k = n/2$, which is possible only if n is even. Say, $n = 2m$. Then placing $n = 2m$ and $k = n/2 = m$ in equation (4), we get for all $m > 1$:

$$\begin{aligned}
F_{2m} &= (F_m)F_{m-1} + (F_{m+1})F_m \\
&\quad \{\text{using recurrence definition of } F_{m+1}\} \\
&= (F_m)(F_{m+1} - F_m) + (F_{m+1})F_m \\
&= 2F_mF_{m+1} - {F_m}^2 \qquad (5)
\end{aligned}$$

Now, to make k and $n-k-1$ as equal, we need $k = (n-1)/2$, which is possible only if n is odd. Say, $n = 2m+1$. Then placing $n = 2m+1$ and $k = (n-1)/2 = m$ in equation (4), we get for all $m \geq 1$:

$$\begin{aligned}
F_{2m+1} &= (F_m)F_m + (F_{m+1})F_{m+1} \\
&= {F_m}^2 + {F_{m+1}}^2 \qquad (6)
\end{aligned}$$

Thus for n being even or odd, we were able to express F_n in terms of two smaller fibonacci numbers which occur almost halfway earlier in the fibonacci sequence.

Example: for $n = 11$ and so $m = 5$, $F_{11} = 89 = {F_5}^2 + {F_6}^2 = 5^2 + 8^2$

Relation 5

We now prove the Cassini's Identity, which states that $\forall n \geq 2$:

$$F_{n+1}F_{n-1} - {F_n}^2 = (-1)^n \qquad (7)$$

We apply Mathematical Induction over n. For $n = 2$:

$$F_{n+1}F_{n-1} - {F_n}^2 = F_3F_1 - {F_2}^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2.$$

So, the identity is true for $n = 2$. Say, it is true for some $n = k$, $k \geq 2$. Then,

$$F_{k+1}F_{k-1} - {F_k}^2 = (-1)^k$$

Now we need to prove it for $n = k + 1$:

$$
\begin{aligned}
F_{n+1}F_{n-1} - {F_n}^2 &= F_{k+2}F_k - {F_{k+1}}^2 \\
&\quad \{\text{expanding } F_{k+2}\text{, and one } F_{k+1} \text{ in} \\
&\quad\ \text{the square term}\} \\
&= (F_k + F_{k+1})F_k - F_{k+1}(F_{k-1} + F_k) \\
&= {F_k}^2 + F_{k+1}F_k - F_{k+1}F_{k-1} - F_{k+1}F_k \\
&= {F_k}^2 - F_{k+1}F_{k-1} \\
&= (-1)(F_{k+1}F_{k-1} - {F_k}^2) \\
&\quad \{\text{induction hypothesis for } n = k\} \\
&= (-1)(-1)^k \\
&= (-1)^{k+1}
\end{aligned}
$$

Thus, if the identity is true for $n = k$, it is also true for $n = k+1$. Hence, the identity must hold for all $n \geq 2$.

Example: for $n = 6$, $F_7F_5 - {F_6}^2 = 13 \cdot 5 - 8^2 = 1 = (-1)^6$

Method of Proving

In Relations 1, 2 and 3, we started with a simple term of fibonacci number(s), and repeatedly used the recurrence definition of fibonacci numbers until we observed a pattern. Thus, we could see the particular relation emerging. Relation 4 came up after an attempt to converge the four different terms of Relation 3.

In Relation 5, we first learned the identity in its full. And then for proving it, we used Mathematical Induction. All other relations can also be proven by Mathematical Induction: Relation 1 and 2 by induction over n, Relation 3 by induction over k for any fixed n.