Multiples of Golden-Ratio “Modulo” 1

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Consider the multiples of the Golden-Ratio ϕ: $\phi, 2\phi, 3\phi, \ldots$. What can we say about the distribution of their fractional-parts in interval [0,1]? In this article we find out how, due to the *Three Distance Theorem* and properties of the Golden-Ratio, these values are well-distributed.

Let $\alpha < 1$ be any positive real number and N any positive integer. For any real number x, let $\{x\}$ denote the fractional part of x, i.e. $\{x\} = x - \lfloor x \rfloor$. What can we say about distribution of values:

$$\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\} \tag{1}$$

in the interval [0,1]?

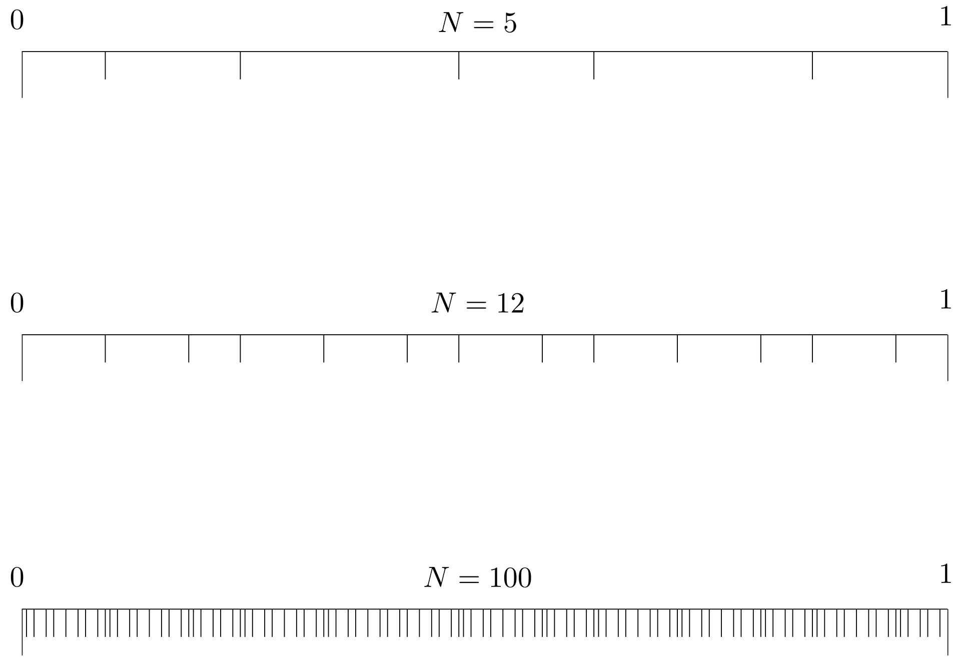
In an earlier article titled *Three Distance Theorem* [1], we analyzed such distribution for any general α. The basic statement of the Three Distance Theorem is very simple. It says that, if the values in (1) are sorted and we find the differences of all neighboring values (including interval boundaries 0 and 1), which we can call “gap lengths”, then there are either two or three distinct gap-lengths. We also learned how these gap-lengths are related to the *Simple Continued Fraction* of α. In this article we will make use of the observations and theorems from [1], for the case of $\alpha = \phi$.

We will often refer the values in (1) as $\{m\alpha\}$ where m is a positive integer. The Simple Continued Fraction will be referred as “Continued Fraction”.

For any positive integers m and n, $\{m(\alpha + n)\} = \{m\alpha\}$. This is the reason that, in [1], it was sufficient to consider only $\alpha < 1$. For the same reason, our analysis here for $\phi \approx 1.6180339\ldots$ also applies exactly to $\phi - 1, \phi + 1, \phi + 2$ etc.

Figure 1 shows the values of $\{m\phi\}$ for different values of N. Due to the Three Distance Theorem, we expect to see either two or three distinct gap-lengths.

Figure 1: $\{m\phi\}$ for some N



Continued Fraction of ϕ

We first note that ϕ is the positive solution to the quadratic equation:

$$\frac{x}{1} = \frac{1+x}{x}$$
$$\Leftrightarrow \quad x^2 - x - 1 = 0$$

So, $\phi = (1+\sqrt{5})/2$, and

$$\phi = 1 + \frac{1}{\phi}$$

Please refer to the conventions used in section "Background" of [1] about continued fractions. The continued fraction of ϕ is:

$$\phi = 1 + \cfrac{1}{1 + \frac{1}{1+\dots}} = [1; 1, 1, 1, \dots]$$

which is special since all a_i terms are 1.

We define $p_{-1} = 1$ and $q_{-1} = 0$. For this continued fraction, we have $p_0 = 1, q_0 = 1$. And due to the recurrence-relation (2) in [1], for all $k \geq 0$:

$$p_{k+1} = a_{k+1} p_k + p_{k-1} = p_k + p_{k-1}$$
$$q_{k+1} = a_{k+1} q_k + q_{k-1} = q_k + q_{k-1}$$

Thus, for this special case of ϕ, the sequences of p_k and q_k values each follow the same recurrence-relation as the Fibonacci Sequence, which is defined as:

$$\begin{aligned} &F_0 = 0 \\ &F_1 = 1 \\ &F_n = F_{n-2} + F_{n-1} \quad \text{for all } n \geq 2 \end{aligned}$$

So, in the continued fraction of ϕ, for all $k \geq 0$:

$$\begin{aligned} p_k &= F_{k+2} \\ q_k &= F_{k+1} \end{aligned}$$

Thus the convergents which approximate ϕ are: $p_k/q_k = F_{k+2}/F_{k+1}$.

Now, consider any integer $N \geq 1$ and its corresponding unique integers t, r and s, as defined by relations (3) and (4) in [1]. The relation (3) in [1], which defines t for N, now becomes:

$$\begin{aligned} & q_{t-1} + q_t \leq N < q_t + q_{t+1} \\ \Leftrightarrow \quad & F_t + F_{t+1} \leq N < F_{t+1} + F_{t+2} \\ \Leftrightarrow \quad & F_{t+2} \leq N < F_{t+3} \end{aligned} \tag{2}$$

So, with $\alpha = \phi$, the $t \geq 0$ corresponding to N is such that F_{t+2} is the largest Fibonacci number not exceeding N.

The range of r as defined by relation (4) in [1] is: $1 \leq r \leq a_{t+1}$. With ϕ, $a_{t+1} = 1$ for all $t \geq 0$. So in this case we have $r = 1$ for all N.

The Three Gap-Lengths

We will be using a few facts from the theory of continued fractions. Say, p_k/q_k represent the convergents from the continued fraction of a real number $\alpha > 0$. Then, it is known that for all $k \geq 0$:

$$\begin{aligned} & \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} \\ \Leftrightarrow \quad & |q_k \alpha - p_k| < \frac{1}{q_{k+1}} \end{aligned}$$

For all $k \geq 0$, since $q_{k+1} \geq 1$, so:

$$|q_k \alpha - p_k| < 1$$

It is also known that, for all $k \geq 0$:

$$\begin{aligned}
& \alpha - \frac{p_k}{q_k} = \left|\alpha - \frac{p_k}{q_k}\right| (-1)^k \\
\Leftrightarrow \quad & q_k\alpha - p_k = |q_k\alpha - p_k|(-1)^k
\end{aligned}$$

Due to above relations, and the fact that any integer can be added or subtracted inside $\{\}$, $\{q_k\alpha\}$ can also be written as:

$$\begin{aligned}
\{q_k\alpha\} &= \{q_k\alpha - p_k\} \\
&= \{1 + q_k\alpha - p_k\} \\
&= \{1 + |q_k\alpha - p_k|(-1)^k\} \\
&= q_k\alpha - p_k & \text{(for even } k\text{)} \\
&\quad \{1 - (p_k - q_k\alpha)\} = 1 - (p_k - q_k\alpha) & \text{(for odd } k\text{)} \quad (3)
\end{aligned}$$

Now we compute the gap-lengths for ϕ. As defined in section "Three Distance Theorem" of [1], we will use u_1 and u_N to refer to the two "corner points".

Consider the case when t is even. Using theorem 4 in [1], the u_1 and u_N for ϕ will be:

$$\begin{aligned}
u_1 &= q_t = F_{t+1} \\
u_N &= q_{t-1} + rq_t = F_t + (1)F_{t+1} = F_{t+2}
\end{aligned}$$

We noted earlier, due to relation (2), that F_{t+2} is the largest Fibonacci number not exceeding N. Now we see that the two corner points correspond to the two largest Fibonacci numbers not exceeding N.

Now we use theorem 2 in [1] to find the three gap-lengths, which are specified as L_1, L_2 and $L_1 + L_2$ in that theorem.

$$\begin{aligned}
L_1 &= \{u_1\alpha\} \\
&= \{q_t\alpha\} \\
&= q_t\alpha - p_t & \text{(due to (3) with } t \text{ even)} \\
&= F_{t+1}\phi - F_{t+2}
\end{aligned}$$

$$\begin{aligned}
L_2 &= 1 - \{u_N\alpha\} \\
&= 1 - \{q_{t+1}\alpha\} & \text{(since } F_{t+2} \text{ is also } q_{t+1}\text{)} \\
&= 1 - (1 - (p_{t+1} - q_{t+1}\alpha)) & \text{(due to (3) with } t+1 \text{ odd)} \\
&= p_{t+1} - q_{t+1}\alpha \\
&= F_{t+3} - F_{t+2}\phi
\end{aligned}$$

$$L_1 + L_2 = F_{t+1}\phi - F_{t+2} + F_{t+3} - F_{t+2}\phi = F_{t+1} - F_t\phi$$

This was for even t. Similarly, it can be proved that for t as odd, the expressions of the three lengths will be:

$$\begin{aligned} L_1 &= -(F_{t+3} - F_{t+2}\phi) \\ L_2 &= -(F_{t+1}\phi - F_{t+2}) \\ L_1 + L_2 &= -(F_{t+1} - F_t\phi) \end{aligned}$$

Corollary 1. *The two smallest gap-lengths (L_1 and L_2, in this order or reverse) for $\alpha = \phi$ are:*

$$|F_{t+2}\phi - F_{t+3}| \text{ and } |F_{t+1}\phi - F_{t+2}|$$

where, F_{t+2} $(t \geq 0)$ is the largest Fibonacci number not exceeding N. The gap-length $L_1 + L_2$ can also be written as:

$$|F_t\phi - F_{t+1}|.$$

Note how u_1 and u_N, and hence the three gap-lengths, change only when t changes. Due to the relation (2) which defines t, that occurs whenever N attains any Fibonacci number.

Note that, gaps of length $L_1 + L_2$ need not exist for all N. From corollary 5 in [1], gap-length $L_1 + L_2$ will not exist if $s = q_t - 1$. For case of ϕ, that can be rewritten as:

$$\begin{aligned} & & s &= q_t - 1 \\ &\Leftrightarrow & N - q_{t-1} - rq_t &= q_t - 1 \qquad \text{(due to relation (4) in [1])} \\ &\Leftrightarrow & N - F_t - F_{t+1} &= F_{t+1} - 1 \\ &\Leftrightarrow & N &= F_{t+3} - 1 \end{aligned}$$

Corollary 2. *For $\alpha = \phi$, the gap-length of $L_1 + L_2$ does not exist whenever $N + 1$ is some Fibonacci number.*

So, in Figure 1 for $N = 12$, which is $F_7 - 1$, we should expect only two distinct gap-lengths.

Alternate Form

For any integer $n > 0$, the expression $F_n\phi - F_{n+1}$ can be repeatedly expanded as below:

$$
\begin{aligned}
F_n\phi - F_{n+1} &= F_n\phi - F_{n-1} - F_n \\
&= F_n(\phi - 1) - F_{n-1} \\
&= \frac{F_n}{\phi} - F_{n-1} && \text{(since } \phi - 1 = 1/\phi\text{)} \\
&= \left(-\frac{1}{\phi}\right)(F_{n-1}\phi - F_n) && \text{(now, repeatedly expand as above)} \\
&= \left(-\frac{1}{\phi}\right)^2 (F_{n-2}\phi - F_{n-1}) \\
&= \left(-\frac{1}{\phi}\right)^3 (F_{n-3}\phi - F_{n-2}) \\
&\dots \\
&\dots \\
&= \left(-\frac{1}{\phi}\right)^n (F_0\phi - F_1) \\
&= -\left(-\frac{1}{\phi}\right)^n && (4)
\end{aligned}
$$

So, the three gap-lengths in corollary 1 can also be expressed as:

$$
\begin{aligned}
|F_{t+2}\phi - F_{t+3}| &= \frac{1}{\phi^{t+2}} \\
|F_{t+1}\phi - F_{t+2}| &= \frac{1}{\phi^{t+1}} \\
|F_t\phi - F_{t+1}| &= \frac{1}{\phi^t} && (5)
\end{aligned}
$$

Ratio of the Gap-Lengths

It is evident from relation (5) that the mutual ratio of the gap-lengths is itself the golden-ratio ϕ. This is an interesting and useful property of the golden-ratio.

Note that for any general real number α, although there are only two or three gap-lengths always, but the lengths need not be in a ratio (large to small) close to 1 or 2. But for $\alpha = \phi$, we found that they are always in the ratio $\phi \approx 1.6$. So, the lengths are not very disproportionate with each other.

In fact, from book [2], chapter 6.4 “Hashing”, exercise 9 we learn that — only when $\{\alpha\} = 1/\phi$ or $1/\phi^2$ (note that $1/\phi = \phi - 1$, and $1/\phi^2 = 1 - 1/\phi$), do we have the two smallest gap-lengths in ratio (large to small) not exceeding 2 for any N number of values. For all other α, this ratio will exceed 2 for some N.

It is this property of ϕ which makes it unique among all real numbers with regard to the “closeness” of the gap-lengths. So, if we are looking to make the distribution of $\{m\alpha\}$ in [0,1] as uniform as possible for all N, then $\{\alpha\} = 1/\phi$ or $1/\phi^2$ are good choices.

In Figure 1, we observe that the distribution of $\{m\phi\}$ for $N = 100$ has good uniformity. There are no cluster of values around any point.

■

References

[1] Nitin Verma. *Three Distance Theorem.* `https://mathsanew.com/articles/three_distance_theorem.pdf` (2020).

[2] Donald E. Knuth. *The Art of Computer Programming*, Vol.3, Second Edition. Addison-Wesley, 1998.