Symmetry in the Roots of a Quadratic Equation

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One way to write a *Quadratic Equation* is:

$$ax^2 + bx + c = 0 \tag{1}$$

where, a, b, c are known real-numbers with $a \neq 0$, and x is an unknown number. Suppose we know one root, r_1, of this equation. Then,

$$ar_1^2 + br_1 + c = 0 \tag{2}$$

$$\Leftrightarrow \quad r_1\left(r_1 + \frac{b}{a}\right) = -\frac{c}{a} \tag{3}$$

So, there are two terms on LHS, r_1 and $(r_1 + b/a)$, which multiply to give a constant $-c/a$ on RHS. Just by looking at this relation, can we know anything about another root r_2 of (1)? Finding such r_2 is equivalent to finding a number which can replace r_1 in (3). That is, the two terms r_2 and $(r_2 + b/a)$ must also multiply to give the same constant $-c/a$.

Multiplication operation is commutative. So, one way to get the same result is to exchange the two terms being multiplied in (3), and treat $(r_1 + b/a)$ as r_2 and r_1 as $(r_2 + b/a)$. But that would require:

$$r_2 = \left(r_1 + \frac{b}{a}\right) = \left(r_2 + \frac{b}{a} + \frac{b}{a}\right) \quad \Leftrightarrow \quad \frac{2b}{a} = 0$$

which is possible only if $b = 0$. So this did not work. Let us try again, but this time we not only exchange the two terms but also negate each, which would still give the same result on multiplication. So, now we treat $-(r_1 + b/a)$ as r_2, and $-r_1$ as $(r_2 + b/a)$. This is feasible because it yields:

$$r_2 = -\left(r_1 + \frac{b}{a}\right) = -\left(-r_2 - \frac{b}{a} + \frac{b}{a}\right) = r_2$$

Thus we have found another root $r_2 = -(r_1 + b/a)$. We could arrive at it by using the fact that two multiplication terms can be exchanged, and both negated, to give the same result. The relation can also be expressed as:

$$r_1 + r_2 = -\frac{b}{a}$$

After doing this, we may be curious to look for a way of using the commutativity of addition, like we did for multiplication (and exchange the two terms being added). We can rewrite (2) as below, assuming $r_1 \neq 0$ and dividing both sides by r_1:

$$\begin{aligned} & ar_1 + b + \frac{c}{r_1} = 0 \\ \Leftrightarrow \qquad & r_1 + \frac{c}{ar_1} = -\frac{b}{a} \end{aligned} \tag{4}$$

We now try to find r_2 which can replace r_1 in above (again assuming $r_2 \neq 0$). Note that the LHS adds two terms r_1 and c/ar_1 to give a constant $-b/a$. Can we exchange these two terms and treat c/ar_1 as r_2, and r_1 as c/ar_2 (for this we need $c \neq 0$, because r_1, r_2 are non-zero)? In fact this is feasible, because it yields:

$$r_2 = \frac{c}{ar_1} = \frac{c}{a} \cdot \frac{ar_2}{c} = r_2$$

Thus, again we could find another root $r_2 = c/ar_1$. Though, we arrived at this under some assumptions. The relation can also be expressed as:

$$r_1 r_2 = \frac{c}{a}$$

In both of the exercises above, we could exchange the two terms being multiplied (or added) for one root, to give us the corresponding two terms for another root. This shows us a symmetry between the two roots. Another way to observe this symmetry is through the expressions $r_1 + r_2$ (equaling $-b/a$) and $r_1 r_2$ (equaling c/a), where r_1 and r_2 appear in the same way and so are interchangeable.

Side Note

Above we could establish the relations about sum and product of the roots (though with some assumptions). There are other ways also to arrive at these relations. Equation (2) for r_1, and similar one for r_2, can be combined and re-arranged as:

$$\begin{aligned} & ar_1^2 + br_1 + c = ar_2^2 + br_2 + c \\ \Leftrightarrow \quad & a(r_1^2 - r_2^2) = b(r_2 - r_1) \\ & \{\text{if } r_1 \neq r_2\} \\ \Leftrightarrow \quad & a(r_1 + r_2) = -b \\ \Leftrightarrow \quad & r_1 + r_2 = -\frac{b}{a} \end{aligned}$$

Also, combining equation (4) for r_1 and similar one for r_2:

$$\begin{aligned} & r_1 + \frac{c}{ar_1} = r_2 + \frac{c}{ar_2} \\ \Leftrightarrow \quad & r_1 - r_2 = \frac{c}{a}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \\ & \{\text{if } r_1 \neq r_2\} \\ \Leftrightarrow \quad & 1 = \frac{c}{a}\left(\frac{1}{r_1 r_2}\right) \\ \Leftrightarrow \quad & r_1 r_2 = \frac{c}{a} \end{aligned}$$

But these derivations involved some assumptions (like $r_1 \neq r_2$), and so are not thorough. There is a complete proof of these relations based on the *Factor Theorem* (which itself is a consequence of the *Polynomial Remainder Theorem*). Due to this theorem, if r_1, r_2 are roots of (1), then, $(x - r_1)$ and $(x - r_2)$ are factors of the quadratic polynomial on the LHS of (1). So,

$$\begin{aligned} & ax^2 + bx + c = a(x - r_1)(x - r_2) \\ \Leftrightarrow \quad & ax^2 + bx + c = ax^2 - ax(r_1 + r_2) + a(r_1 r_2) \end{aligned}$$

Equating the coefficients of same powers of x of both sides, we get:

$$\begin{aligned} r_1 + r_2 &= -\frac{b}{a} \\ r_1 r_2 &= \frac{c}{a} \end{aligned}$$

Of course, we can also prove these relations by first deriving the roots to be $(-b \pm \sqrt{b^2 - 4ac})/2a$, and then simplifying $r_1 + r_2$ and $r_1 r_2$.

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