Three Distance Theorem

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Let $\alpha < 1$ be any positive real number and N any positive integer. For any real number x, let $\{x\}$ denote the fractional part of x, i.e. $\{x\} = x - \lfloor x \rfloor$. What can we say about distribution of values:

$$\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\} \tag{1}$$

in the interval [0,1]? Do these values get clustered at some places or they are almost uniformly spread across [0,1]?

In this article, we will try answering the above with help of the *Three Distance Theorem*. The theorem is also known as the *Three Gap Theorem* and *Steinhaus Conjecture* (due to Hugo Steinhaus, who originally conjectured it).

We will often refer the values in (1) as $\{m\alpha\}$ where m is a positive integer. Whenever α is a rational number p/q, it will be assumed that p and q are positive coprime integers and $p < q$.

For any real number $\alpha' > 1$ which is not an integer, we know that there exist integer n and another positive real number $\alpha < 1$ such that $\alpha' = \alpha + n$. So, for any positive integer m, $\{m\alpha'\} = \{m\alpha\}$. Hence, all values $\{m\alpha'\}$ are exactly same as values $\{m\alpha\}$ and we can restrict our analysis to $\alpha < 1$.

The α being a real number can be rational or irrational. Say, α is a rational number p/q. With the help of Modular Arithmetic (refer to Corollary 4 in article titled *Multiples of an Integer Modulo Another Integer*), it can be proved that, for $m = 1, 2, \ldots, q$, the values $\{m\alpha\}$ are all distinct and form the set: $S = \{0, 1/q, 2/q, \ldots, (q-1)/q\}$.

For $m = q + 1$ onward, these $\{m\alpha\}$ values repeat, because for $n = 1, 2, 3, \ldots$, $\{(q+n)(p/q)\} = \{n(p/q)\}$. So for any $N \geq q$, we now know that the values in (1) form the set S and so are distributed uniformly in interval [0,1], spaced by distance $1/q$.

For any $N < q$, we only know that values $\{m\alpha\}$ are some subset of S of size N, but we do not know any further about their distribution. This is where the Three Distance Theorem will help us.

Now, say α is an irrational number. That is, there exist no integer p and q such that $\alpha = p/q$. Do the values $\{m\alpha\}$ repeat in this case also? Assume that they repeat and $\{m_1\alpha\} = \{m_2\alpha\}$ for some positive integers m_1 and m_2 with $m1 < m2$. Then,

$$\begin{aligned} & \{m_1\alpha\} = \{m_2\alpha\} \\ \Leftrightarrow \quad & m_2\alpha - m_1\alpha = \text{an integer, say } n \\ \Leftrightarrow \quad & \alpha(m_2 - m_1) = n \\ \Leftrightarrow \quad & \alpha = n/(m_2 - m_1) \end{aligned}$$

But this would mean α is a rational number, a contradiction.

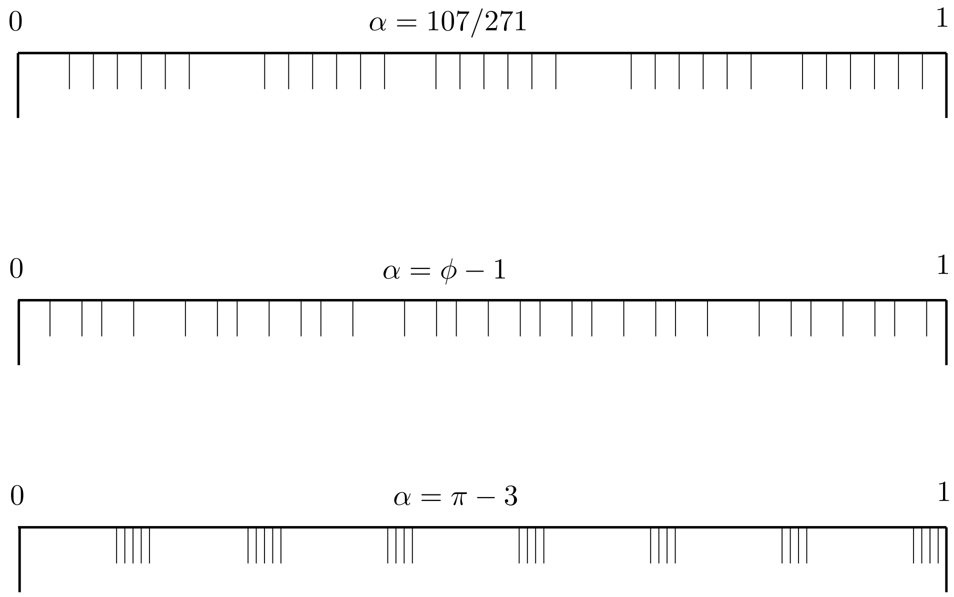
Thus we arrive at an interesting conclusion that when α is irrational, values $\{m\alpha\}$ are distinct for all positive integers m. Similarly we can prove that when α is irrational, there is no positive integer m such that $\{m\alpha\} = 0$.

The basic statement of the Three Distance Theorem is very simple. It says that, if the values in (1) are sorted and we find the differences of all neighboring values (including interval boundaries 0 and 1), which we can call "gap lengths", then there are either two or three distinct gap-lengths. The theorem holds for all α and N specified in (1), except that when α is rational p/q, then it only holds for $N < q$.

When $\alpha = p/q$, then as discussed above, for all $N \geq q$, there will be only one distinct gap-length of $1/q$. Throughout this discussion, we assume that $N < q$ whenever α is rational p/q.

The figure 1 below shows $N = 30$ points for α as rational $107/271$, as irrational $\phi - 1$ (ϕ is the Golden Ratio) and irrational $\pi - 3$ ($\phi \approx 1.6180339\ldots, \pi \approx 3.14159\ldots$). As per the theorem, for each case, there must be two or three distinct gap-lengths between neighboring points, including 0 and 1. We only took $\alpha < 1$, but as noted earlier, the $\{m\alpha\}$ values will be same if we add a positive integer to the α.

Figure 1: Some α with $N = 30$



There are further details in this theorem as we will see below. It also makes use of the concept of *Continued Fractions*. So we will first review the necessary background. The theorem's statements and some of its details provided below are based on [1] and [2].

Background

For any positive real number α, its *Simple Continued Fraction* can be written as:

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

where for all $i > 0$, a_i are positive integers, and a_0 is a non-negative integer.

If α is irrational, this continued fraction never terminates, i.e. contains infinitely many a_i terms, and is denoted by: $[a_0; a_1, a_2, \ldots]$. For rational α, it always terminates at some term a_n, and is denoted by: $[a_0; a_1, a_2, \ldots, a_n]$.

For example, $25/11 = [2; 3, 1, 2], 14/9 = [1; 1, 1, 4]$, the Golden Ratio ϕ (irrational) $= [1; 1, 1, 1, \ldots], \pi$ (irrational) $= [3; 7, 15, 1, 292, \ldots]$.

We may consider only the first few a_i terms as below and call it the k^{th} *Convergent*: $p_k/q_k = [a_0; a_1, a_2, \ldots, a_k]$. For example,

$$p_0/q_0 = a_0$$
$$p_1/q_1 = a_0 + 1/a_1$$
$$p_2/q_2 = a_0 + 1/(a_1 + 1/a_2)$$

If we define $p_{-1} = 1, q_{-1} = 0$, the following recurrence relations hold for all $k \geq 0$:

$$\begin{aligned} p_{k+1} &= a_{k+1}p_k + p_{k-1} \\ q_{k+1} &= a_{k+1}q_k + q_{k-1} \end{aligned} \tag{2}$$

It is known that p_k and q_k are coprime for all k.

The Three Distance Theorem also involves use of integer values $(q_{k-1} + q_k)$ for $k \geq 0$. Note that the sequence of q_k values is: $q_{-1} = 0, q_0 = 1, q_1 = a_1, q_2 = a_2a_1 + 1$ etc. From (2) we also know the recurrence relation of q_k. Since $a_i \geq 1$ for all $i > 0$, q_k are strictly increasing with k for $k \geq 1$. We also see that the sequence of $(q_{k-1} + q_k)$ is strictly increasing with k for all $k \geq 0$, starting at 1 for $k = 0$.

Hence, any integer $N \geq 1$ either equals $(q_{k-1} + q_k)$ for some $k \geq 0$, or is between two consecutive values of this sequence. We can conclude that, any integer $N \geq 1$ corresponds to a unique integer $t \geq 0$ such that:

$$q_{t-1} + q_t \leq N < q_t + q_{t+1} \tag{3}$$

Note that the last endpoint of this range can be written as:

$$\begin{aligned} q_t + q_{t+1} - 1 &= q_t + (a_{t+1}q_t + q_{t-1}) - 1 \\ &= q_{t-1} + (a_{t+1})q_t + (q_t - 1) \end{aligned}$$

So, in the range of (3), $(N - q_{t-1})$ varies from q_t to $(a_{t+1})q_t + (q_t - 1)$. Due to the Division Theorem, given $(N - q_{t-1})$ and q_t, there exist unique integers r and s such that:

$$\begin{aligned} &(N - q_{t-1}) = rq_t + s && (4) \\ &0 \leq s \leq q_t - 1 && \{\text{Division Theorem}\} \\ &1 \leq r \leq a_{t+1} && \{\text{due to the range of } (N - q_{t-1})\} \end{aligned}$$

Note that for a fixed t, as N varies in range of (3), every increase in N by q_t would increase r by 1.

Thus we now know that, any integer $N \geq 1$ corresponds to unique integers t, r and s as defined by (3) and (4).

Let us now move to exploring the theorem.

Three Distance Theorem

Let $\alpha < 1$ be any positive real number and N any positive integer. As said earlier, we assume that $N < q$ whenever α is rational p/q. Now consider the increasing order of the N values $\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$, which must be all distinct (as concluded on page 2). Say $u_1, u_2, \ldots, u_N$ denote this order such that $\{u_1, u_2, \ldots, u_N\} = \{1, 2, \ldots, N\}$ and, $\{u_i\alpha\} < \{u_{i+1}\alpha\}$ for $i = 1, 2, \ldots, N-1$. So the lowest and highest values are $\{u_1\alpha\}$ and $\{u_N\alpha\}$ respectively.

The basic statement of the theorem is as below.

Theorem 1 (Three Distance Theorem)**.** *For all $i = 1, 2, \ldots, N-1$,*

$$u_{i+1} - u_i = \begin{cases} u_1, & 1 \leq u_i < N + 1 - u_1 \quad (a) \\ u_1 - u_N, & N + 1 - u_1 \leq u_i < u_N \quad (b) \\ -u_N, & u_N \leq u_i < N + 1 \quad (c) \end{cases}$$

The points $\{u_i\alpha\}$ and $\{u_{i+1}\alpha\}$ are neighbors. The theorem says that for any pair of integers u_i and u_{i+1}, their difference can only be one among the three values: $u_1, u_1 - u_N, -u_N$.

Gap Lengths

For $i = 1, 2, \ldots, N-1$, let us refer to the distance from $\{u_i\alpha\}$ to $\{u_{i+1}\alpha\}$ as *gap-i*. The distance from 0 to $\{u_1\alpha\}$ can be referred as *gap*-0, and that from $\{u_N\alpha\}$ to 1 can be referred as *gap-N*. Trivially, *gap*-0 and *gap-N* are $\{u_1\alpha\}$ and $1 - \{u_N\alpha\}$ respectively.

Consider some *gap-i* such that u_i lies in interval (b) of theorem 1. Then,

$$\begin{aligned}
\{u_{i+1}\alpha\} &= \{(u_i + u_1 - u_N)\alpha\} \\
&= \{u_i\alpha + u_1\alpha - u_N\alpha\} \\
&\quad \text{(from all the three real numbers inside \{\}, we can} \\
&\quad \text{ remove their integer part, and add 1 for the} \\
&\quad \text{ negative number)} \\
&= \{\{u_i\alpha\} + \{u_1\alpha\} + 1 - \{u_N\alpha\}\}
\end{aligned}$$

Say, $f = \{u_1\alpha\} + 1 - \{u_N\alpha\}$, which is the sum of *gap*-0 and *gap*-N. So, $0 < f \leq 1$. Then,

$$\begin{aligned}
\{u_{i+1}\alpha\} &= \{\{u_i\alpha\} + f\} \\
&\quad \text{(since } 0 < \{u_i\alpha\} < 1\text{, so, } 0 < \{u_i\alpha\} + f < 2\text{)} \\
&= \{u_i\alpha\} + f\text{, or, } \{u_i\alpha\} + f - 1 \\
&\quad \text{(since } \{u_i\alpha\} + f - 1 \leq \{u_i\alpha\}\text{, but } \{u_{i+1}\alpha\} > \{u_i\alpha\}\text{)} \\
&= \{u_i\alpha\} + f \\
&= \{u_i\alpha\} + \{u_1\alpha\} + 1 - \{u_N\alpha\}
\end{aligned}$$

Thus, in case when u_i lies in interval (b), *gap-i* is $\{u_{i+1}\alpha\} - \{u_i\alpha\} = \{u_1\alpha\} + 1 - \{u_N\alpha\}$. Similarly, we can prove that when u_i lies in intervals (a) or (c), *gap-i* is $\{u_1\alpha\}$ and $1 - \{u_N\alpha\}$ respectively.

Note that u_N, which corresponds to *gap*-N, also falls into the range of interval (c). Now we can conclude the following about all of the $N+1$ gaps, including *gap*-0 and *gap*-N. We count the number of u_i integers in each of the three intervals.

Theorem 2. *Among the total $N+1$ gaps, there are $N+1-u_1$ gaps of length $L_1 = \{u_1\alpha\}$, $N+1-u_N$ gaps of length $L_2 = 1-\{u_N\alpha\}$, and $u_N+u_1-(N+1)$ gaps of length L_1+L_2. The gaps of length L_1+L_2 exist iff $u_N+u_1 > N+1$.*

Can the gap-lengths L_1 and L_2 ever become equal? It can be proved that they will always be distinct for irrational α. Assuming they are equal:

$$\begin{aligned}
& L_1 = L_2 \\
\Leftrightarrow\quad & \{u_1\alpha\} = 1 - \{u_N\alpha\} \\
\Leftrightarrow\quad & \{u_1\alpha\} + \{u_N\alpha\} = 1 \\
\Leftrightarrow\quad & u_1\alpha + u_N\alpha = \text{an integer, say } n \\
\Leftrightarrow\quad & \alpha = n/(u_1 + u_N)
\end{aligned}$$

But this would mean α is a rational number, a contradiction.

Corollary 3. *For irrational α, the gap-lengths L_1 and L_2 are never equal. So, when gaps of length $L_1 + L_2$ exist, there are three distinct gap-lengths, otherwise two distinct gap-lengths.*

We have seen how the length of all gaps are related to u_1 and u_N. The next section presents a theorem which relates u_1 and u_N to the Continued Fraction of α.

u_1, u_N and Continued Fraction of α

Theorem 4. *For integer $N \geq 1$, let t, r and s be the unique integers as defined by* (3) *and* (4)*. Then,*

$$\begin{array}{lll} \text{if } t \text{ is even,} & u_1 = q_t, & u_N = q_{t-1} + rq_t \\ \text{otherwise,} & u_1 = q_{t-1} + rq_t, & u_N = q_t \end{array}$$

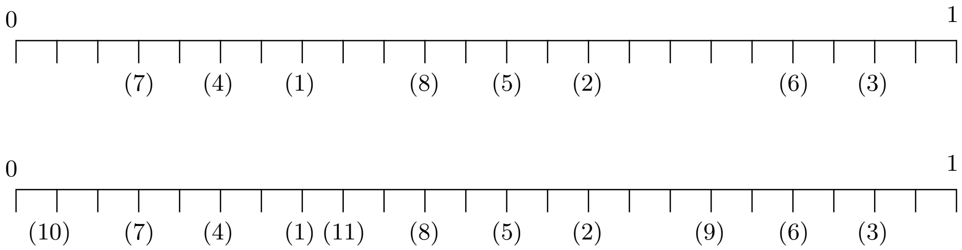
Given α and N, we can use theorem 4 to find u_1 and u_N from the continued fraction of α and then use theorem 2 to find the gap-lengths.

Let us see an example of how the gap-lengths get created with $\alpha = 7/23$. Its simple continued fraction is: $7/23 = 0 + 1/(3 + 1/(3 + 1/2)) = [0; 3, 3, 2]$.

So the convergents are: $p_0/q_0 = 0/1, p_1/q_1 = 1/3, p_2/q_2 = 3/10, p_3/q_3 = 7/23$.

The figure 2 below shows $N = 8$ and $N = 11$ points. The value $\{m\alpha\}$ is indicated with (m), $m = 1, 2, \ldots, N$. Since all values $\{m\alpha\} = \{7m/23\}$ will be some multiples of $1/23$, so in the figure we have also marked all multiples of $1/23$ in [0,1].

Figure 2: $\alpha = 7/23$



Since $q_0 + q_1 = 4$ and $q_1 + q_2 = 13$, so for both $N = 8$ and $N = 11$, $t = 1$ based on (3). Using (4) with $t = 1$,

$$N - q_{t-1} = rq_t + s \quad \Leftrightarrow \quad N - 1 = 3r + s$$

So, for $N = 8$, $r = 2$ and $s = 1$. For $N = 11$, $r = 3$ and $s = 1$.

Using theorem 4, as t is odd, so for $N = 8$:

$$u_1 = rq_t + q_{t-1} = 2(3) + 1 = 7, u_N = q_t = 3$$

and for $N = 11$:

$$u_1 = rq_t + q_{t-1} = 3(3) + 1 = 10, u_N = q_t = 3$$

So for both N, the gap-length $L_2 = 1 - \{u_N\alpha\} = 1 - \{3(7/23)\} = 2/23$. For $N = 8$, gap-length $L_1 = \{u_1\alpha\} = \{7(7/23)\} = 3/23$. For $N = 11$, gap-length $L_1 = \{u_1\alpha\} = \{10(7/23)\} = 1/23$.

The gap-length $L_1 + L_2$, for $N = 8$ is $3/23 + 2/23 = 5/23$, and for $N = 11$ is $1/23 + 2/23 = 3/23$.

Using theorem 2, the counts of gap-length L_1, L_2 and $L_1 + L_2$ must respectively be: $N + 1 - u_1$, $N + 1 - u_N$ and $u_1 + u_N - (N + 1)$. For $N = 8$, these counts are: 2, 6 and 1. For $N = 11$, these counts are: 2, 9 and 1.

We see these same gap-lengths and their counts in the figure also.

We can also relate the existence of $L_1 + L_2$ gap-length with q_t and s as below.

Corollary 5. *The gap-length of $L_1 + L_2$ exists iff $s < q_t - 1$.*

Proof. From theorem 2, gap-length of $L_1 + L_2$ exist *iff* :

$$\begin{aligned}
& u_N + u_1 > N + 1 \\
\Leftrightarrow \quad & q_t + q_{t-1} + rq_t > N + 1 && \{\text{using theorem 4}\} \\
\Leftrightarrow \quad & q_t + q_{t-1} + rq_t > (rq_t + s + q_{t-1}) + 1 && \{\text{using (4)}\} \\
\Leftrightarrow \quad & q_t - 1 > s
\end{aligned}$$

□

As the range of s is $0 \leq s \leq q_t - 1$, so this gap-length does not exist when $s = q_t - 1$.

In the above example with $\alpha = 7/23$, the gap-length $L_1 + L_2$ existed for both N because for both N, $s = 1$ and so $q_t - 1 = 2 > s$.

Some Elaborations About Theorem 4

$\underline{N < 1 + q_1}$ In (3), note the particular case of $t = 0$, i.e.

$$q_{-1} + q_0 \leq N < q_0 + q_1 \quad \Leftrightarrow \quad 1 \leq N < 1 + q_1$$

Using (4) with $t = 0$,

$$N - q_{t-1} = rq_t + s \quad \Leftrightarrow \quad N = r(1) + s$$

So for this range of N, r always equals N and s always equals 0. The two corner points, $u_1 = q_t = 1$, $u_N = q_{t-1} + rq_t = r = N$, simply remain 1 and N for all such N. Due to Corollary 5, the gap-length $L_1 + L_2$ does not exist because $s = 0 = q_t - 1$.

A simple example of this case is with $\alpha = 2/11 = [0; 5, 2]$, where $q_1 = 5$. So for all $N < 1 + q_1 = 6$, the corner points are 1 and N, and only two gap-lengths will exist.

$\underline{N = q_{t+1}}$ The below range of N which decides t (from (3)):

$$q_{t-1} + q_t \leq N < q_t + q_{t+1}$$

can be split into two subranges:

$$(a) \quad q_{t-1} + q_t \leq N < q_{t-1} + a_{t+1}q_t = q_{t+1}$$
$$(b) \quad q_{t+1} \leq N < q_t + q_{t+1}$$

Refer to relation (4). In (a), $(N - q_{t-1})$ ranges from q_t to $a_{t+1}q_t - 1$. So, r would take values 1 to $a_{t+1} - 1$. For all N in this subrange, the two corner points $((u_1, u_N)$ or $(u_N, u_1))$ will be: q_t and $q_{t-1} + rq_t$.

In (b), $(N - q_{t-1})$ ranges from $a_{t+1}q_t$ to $a_{t+1}q_t + (q_t - 1)$. So, r remains fixed at a_{t+1}, and s varies from 0 to $q_t - 1$. The two corner points remain fixed at: q_t and $(q_{t-1} + a_{t+1}q_t = q_{t+1})$. It is this q_{t+1} corner point which continues to remain the corner point when N crosses this range and enters into a new range with "t" as $t + 1$.

$\underline{q_t \text{ increments in } N}$ Also observe how the corner points change as N varies in the range for a particular t: $q_{t-1} + q_t \leq N < q_t + q_{t+1}$.

Suppose t is even (odd case is similar) and so $u_1 = q_t$ and $u_N = q_{t-1} + rq_t$. For all N in this range, u_1 remains fixed, but u_N changes whenever r changes. When N starts at $q_{t-1} + q_t$, $r = 1$, so $u_N = q_{t-1} + q_t = N$. Then, for every q_t increment in N, r increments by 1, and so u_N changes too. This also implies simultaneous changes in the gap-lengths due to theorem 2.

■

References

[1] Tony Van Ravenstein. *The Three Gap Theorem (Steinhaus Conjecture)*. J. Austral. Math. Soc. (Series A) 45 (1988), 360-370.

[2] Victor Beresnevich, Nicol Leong. *Sums of reciprocals and the three distance theorem*. http://eprints.whiterose.ac.uk/121745/ (2017).